Fuzzy Logic

Contents

Characteristics of Fuzzy Sets

- The classical set theory developed in the late 19th century by Georg Cantor describes how crisp sets can interact. These interactions are called **operations**.

- Also fuzzy sets have well defined **properties**.

- These properties and operations are the basis on which the fuzzy sets are used to deal with uncertainty on the one hand and to represent knowledge on the other.

Note: Membership Functions

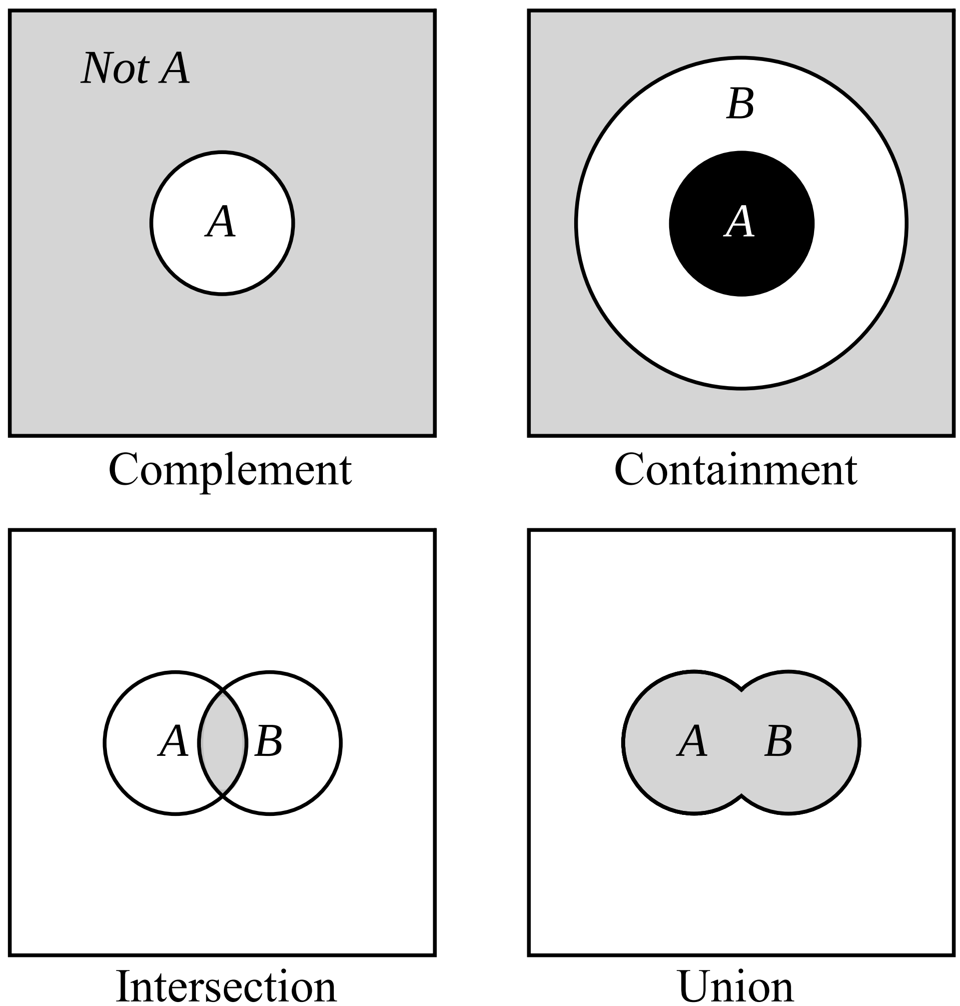
- For the sake of convenience, usually a fuzzy set is denoted as:

$$A = \mu_A(x_i)/x_i + \ldots\ldots\ldots\ldots + \mu_A(x_n)/x_n$$

where $\mu_A(x_i)/x_i$ (a singleton) is a pair “grade of membership” element, that belongs to a finite universe of discourse:

$$A = \{x_1, x_2, .., x_n\}$$

Operations of Fuzzy Sets



Complement

- Crisp Sets: Who does not belong to the set?
- Fuzzy Sets: How much do elements not belong to the set?

- The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. When we remove the tall men set from the universe of discourse, we obtain the complement.

- If A is the fuzzy set, its complement $\neg A$ can be found as follows:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

Containment

- Crisp Sets: Which sets belong to which other sets?
- Fuzzy Sets: Which sets belong to other sets?

- Similar to a Chinese box, a set can contain other sets. The smaller set is called the **subset**. For example, the set of tall men contains all tall men; very tall men is a subset of tall men. However, the tall men set is just a subset of the set of men. In crisp sets, all elements of a subset entirely belong to a larger set. In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in it than in the larger set.

Intersection

- Crisp Sets: Which element belongs to both sets?
- Fuzzy Sets: How much of the element is in both sets?

- In classical set theory, an intersection between two sets contains the elements shared by these sets. For example, the intersection of the set of tall men and the set of fat men is the area where these sets overlap. In fuzzy sets, an element may partly belong to both sets with different memberships.

- A fuzzy intersection is the **lower membership** in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on universe of discourse X:

$$\mu_{A \cap B}(x) = \min\,[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x),$$

where $x \in X$

Union

- Crisp Sets: Which element belongs to either set?
- Fuzzy Sets: How much of the element is in either set?

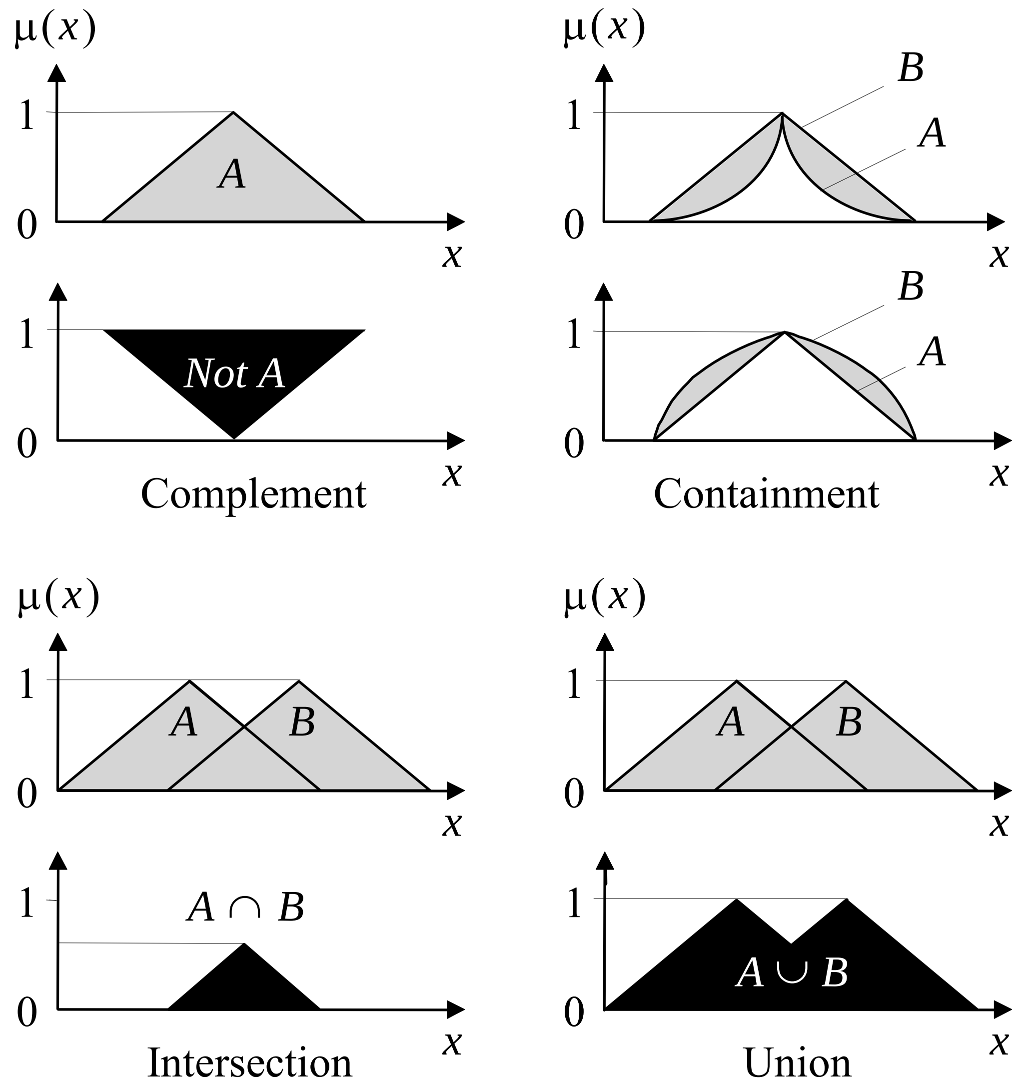
- The union of two crisp sets consists of every element that falls into either set. For example, the union of tall men and fat men contains all men who are tall OR fat.

- In fuzzy sets, the union is the reverse of the intersection. That is, the union is the **largest membership** value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:

$$\mu A \cup B(x) = \max [\mu A(x), \mu B(x)] = \mu A(x) \cup \mu B(x),$$

where $x \in X$

Operations of Fuzzy Sets



Properties of Fuzzy Sets

- Equality of two fuzzy sets
- Inclusion of one set into another fuzzy set
- Cardinality of a fuzzy set
- An empty fuzzy set
- α-cuts (alpha-cuts)

Equality

- Fuzzy set A is considered equal to a fuzzy set B, IF AND ONLY IF (iff):

$$\mu_A(x) = \mu_B(x),\ \forall x \in X$$

$A = 0.3/1 + 0.5/2 + 1/3$

$B = 0.3/1 + 0.5/2 + 1/3$

therefore $A = B$

Inclusion

- Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:

  $\mu_A(x) \le \mu_B(x), \forall x \in X$

  Consider $X = \{1, 2, 3\}$ and sets A and B

  $A = 0.3/1 + 0.5/2 + 1/3;$

  $B = 0.5/1 + 0.55/2 + 1/3$

  then A is a subset of B, *or* $A \subseteq B$

Cardinality

- Cardinality of a non-fuzzy set, Z, is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:

$$card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \Sigma\mu_A(x_i), \qquad \text{for } i=1..n$$

Consider $X = \{1, 2, 3\}$ and sets A and B

$A = 0.3/1 + 0.5/2 + 1/3;$

$B = 0.5/1 + 0.55/2 + 1/3$

$card_A = 1.8$

$card_B = 2.05$

Empty Fuzzy Set

- A fuzzy set A is empty, IF AND ONLY IF:

  $\mu_A(x) = 0, \forall x \in X$

  Consider $X = \{1, 2, 3\}$ and set A

  $A = 0/1 + 0/2 + 0/3$

  then A is *empty*

Alpha-cut

- An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, such that:

  $$A_\alpha = \{\mu_A(x) \geq \alpha,\ \forall x \in X\}.$$

  Consider $X = \{1, 2, 3\}$ and set A

  $A = 0.3/1 + 0.5/2 + 1/3$

  then $A_{0.5} = \{2, 3\}$,

  $A_{0.1} = \{1, 2, 3\}$,

  $A_1 = \{3\}$

Fuzzy Set Normality

- A fuzzy subset of X is called **normal** if there exists at least one element $x \in X$ such that $\mu_A(x) = 1$.

- A fuzzy subset that is not normal is called **subnormal**.

- All crisp subsets except for the null set are normal. In fuzzy set theory, the concept of nullness essentially generalises to subnormality.

- The **height** of a fuzzy subset A is the large membership grade of an element in A

  $$height(A) = \max_x(\mu_A(x))$$

Fuzzy Sets Core and Support

- Assume A is a fuzzy subset of X:
- the **support** of A is the crisp subset of X consisting of all elements with membership grade:

$$supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$$

- the **core** of A is the crisp subset of X consisting of all elements with membership grade:

$$core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$$

Fuzzy Set Math Operations

- $aA = \{a\mu_A(x), \forall x \in X\}$

  Let a =0.5, and

  A = {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  A^a = {0.25/a, 0.15/b, 0.1/c, 0.5/d}

- $A^a = \{\mu_A(x)^a, \forall x \in X\}$

  Let a =2, and

  A = {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  A^a = {0.25/a, 0.09/b, 0.04/c, 1/d}

- …

Fuzzy Sets Examples

- Consider two fuzzy subsets of the set *X*,

  X = {a, b, c, d, e }

  referred to as *A* and *B*

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  and

  B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

Fuzzy Sets Examples

- Support:

  $supp(A) = \{a, b, c, d\}$

  $supp(B) = \{a, b, c, d, e\}$

- Core:

  $core(A) = \{a\}$

  $core(B) = \{o\}$

- Cardinality:

  $card(A) = 1+0.3+0.2+0.8+0 = 2.3$

  $card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$

Fuzzy Sets Examples

- Complement:

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  $\neg A$ = {0/a, 0.7/b, 0.8/c 0.2/d, 1/e}

- Union:

  $A \cup B$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e}

- Intersection:

  $A \cap B$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

Fuzzy Sets Examples

- *aA*:

  for a=0.5

  $aA = \{0.5/a, 0.15/b, 0.1/c, 0.4/d, 0/e\}$

- A^a:

  for a=2

  $A^a = \{1/a, 0.09/b, 0.04/c, 0.64/d, 0/e\}$

- *a*-cut:

  $A_{0.2} = \{a, b, c, d\}$

  $A_{0.3} = \{a, b, d\}$

  $A_{0.8} = \{a, d\}$

  $A_1 = \{a\}$

Fuzzy Rules

- In 1973, Lotfi Zadeh published his second most influential paper. This paper outlined a new approach to analysis of complex systems, in which Zadeh suggested capturing human knowledge in **fuzzy rules**.

- A fuzzy rule can be defined as a conditional statement in the form:

IF	x	is A
THEN	y	is B

- where x and y are linguistic variables; and A and B are linguistic values determined by fuzzy sets on the universe of discourses X and Y, respectively.

Classical Vs Fuzzy Rules

- A classical IF-THEN rule uses binary logic, for example,

Rule: 1	Rule: 2
IF speed is > 100	IF speed is < 40
THEN stopping_distance is long	THEN stopping_distance is short

- The variable speed can have any numerical value between 0 and 220 km/h, but the linguistic variable stopping_distance can take either value long or short. In other words, classical rules are expressed in the black-and-white language of Boolean logic.

Classical Vs Fuzzy Rules

- We can also represent the stopping distance rules in a fuzzy form:

Rule: 1	Rule: 2
IF speed is fast	IF speed is slow
THEN stopping_distance is long	THEN stopping_distance is short

- In fuzzy rules, the linguistic variable speed also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as slow, medium and fast. The universe of discourse of the linguistic variable stopping_distance can be between 0 and 300 m and may include such fuzzy sets as short, medium and long.

Classical Vs Fuzzy Rules

- Fuzzy rules relate fuzzy sets.

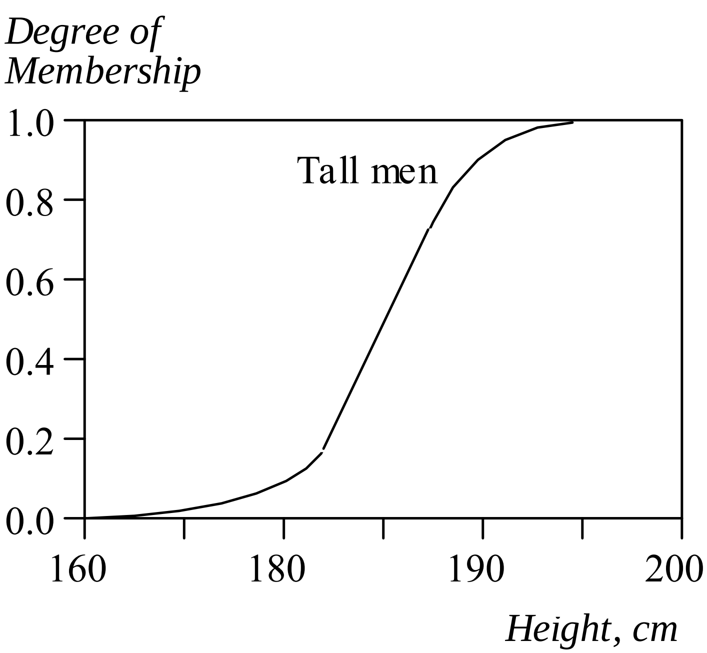
- In a fuzzy system, all rules fire to some extent, or in other words they fire partially. If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

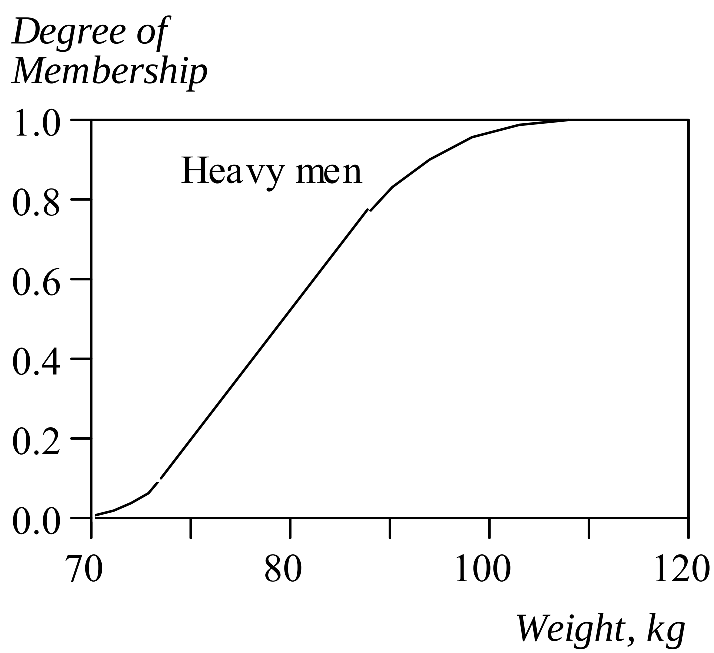
Firing Fuzzy Rules

- These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

  IF height is tall

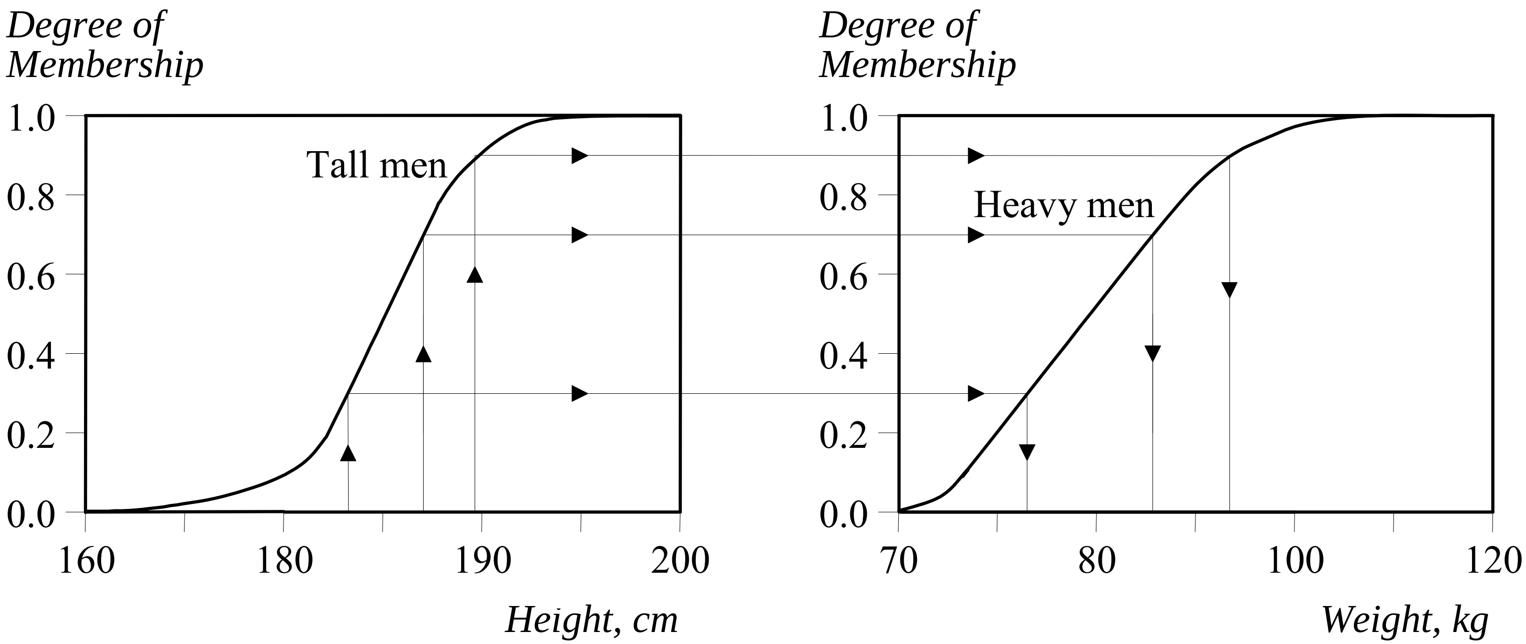
  THEN weight is heavy





Firing Fuzzy Rules

- The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called **monotonic selection**.



Firing Fuzzy Rules

- A fuzzy rule can have multiple antecedents, for example:

IF	project_duration is long
AND	project_staffing is large
AND	project_funding is inadequate
THEN	risk is high

IF	service is excellent
OR	food is delicious
THEN	tip is generous

- The consequent of a fuzzy rule can also include multiple parts, for instance:

IF	temperature is hot
THEN	hot_water is reduced;
	cold_water is increased

Fuzzy Sets Example

- Air-conditioning involves the delivery of air which can be warmed or cooled and have its humidity raised or lowered.

- An air-conditioner is an apparatus for controlling, especially lowering, the temperature and humidity of an enclosed space. An air-conditioner typically has a fan which blows/cools/circulates fresh air and has cooler and the cooler is under thermostatic control. Generally, the amount of air being compressed is proportional to the ambient temperature.

- Consider Johnny’s air-conditioner which has five control switches: COLD, COOL, PLEASANT, WARM and HOT. The corresponding speeds of the motor controlling the fan on the air-conditioner has the graduations: MINIMAL, SLOW, MEDIUM, FAST and BLAST.

Fuzzy Sets Example

- The rules governing the air-conditioner are as follows:

RULE 1:

IF	TEMP is COLD	THEN	SPEED is MINIMAL

RULE 2:

IF	TEMP is COOL	THEN	SPEED is SLOW

RULE 3:

IF	TEMP is PLEASANT	THEN	SPEED is MEDIUM

RULE 4:

IF	TEMP is WARM	THEN	SPEED is FAST

RULE 5:

IF	TEMP is HOT	THEN	SPEED is BLAST

Fuzzy Sets Example

The **temperature** graduations are related to Johnny's perception of ambient temperatures.

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Temp (^{0}C).	**COLD**	**COOL**	**PLEASANT**	**WARM**	**HOT**
0	**Y***	**N**	**N**	**N**	**N**
5	**Y**	**Y**	**N**	**N**	**N**
10	**N**	**Y**	**N**	**N**	**N**
12.5	**N**	**Y***	**N**	**N**	**N**
15	**N**	**Y**	**N**	**N**	**N**
17.5	**N**	**N**	**Y***	**N**	**N**
20	**N**	**N**	**N**	**Y**	**N**
22.5	**N**	**N**	**N**	**Y***	**N**
25	**N**	**N**	**N**	**Y**	**N**
27.5	**N**	**N**	**N**	**N**	**Y**
30	**N**	**N**	**N**	**N**	**Y***

Fuzzy Sets Example

Johnny's perception of the **speed** of the motor is as follows:

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Rev/sec (RPM)	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	N	N	N	N
20	Y	Y	N	N	N
30	N	Y*	N	N	N
40	N	Y	N	N	N
50	N	N	Y*	N	N
60	N	N	N	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the **temperature** are:

- COLD:

  for $0 \leq t \leq 10$ $\mu_{COLD}(t) = - t / 10 + 1$

- SLOW:

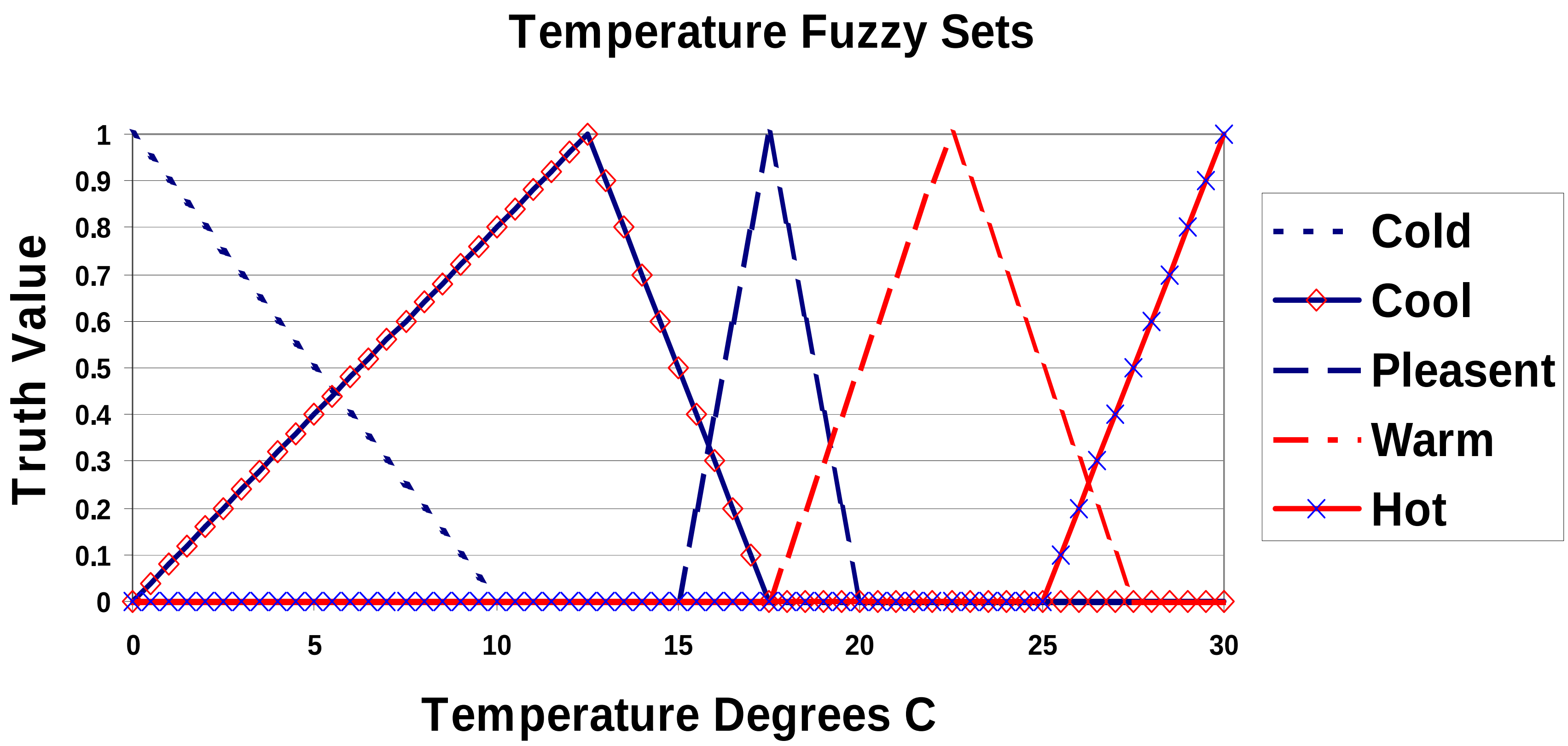
  for $0 \leq t \leq 12.5$ $\mu_{SLOW}(t) = t / 12.5$

  for $12.5 \leq t \leq 17.5$ $\mu_{SLOW}(t) = - t / 5 + 3.5$

- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the **temperature** are:

- MINIMAL:

  for $0 \le v \le 30$ $\mu_{COLD}(t) = -v / 30 + 1$

- SLOW:

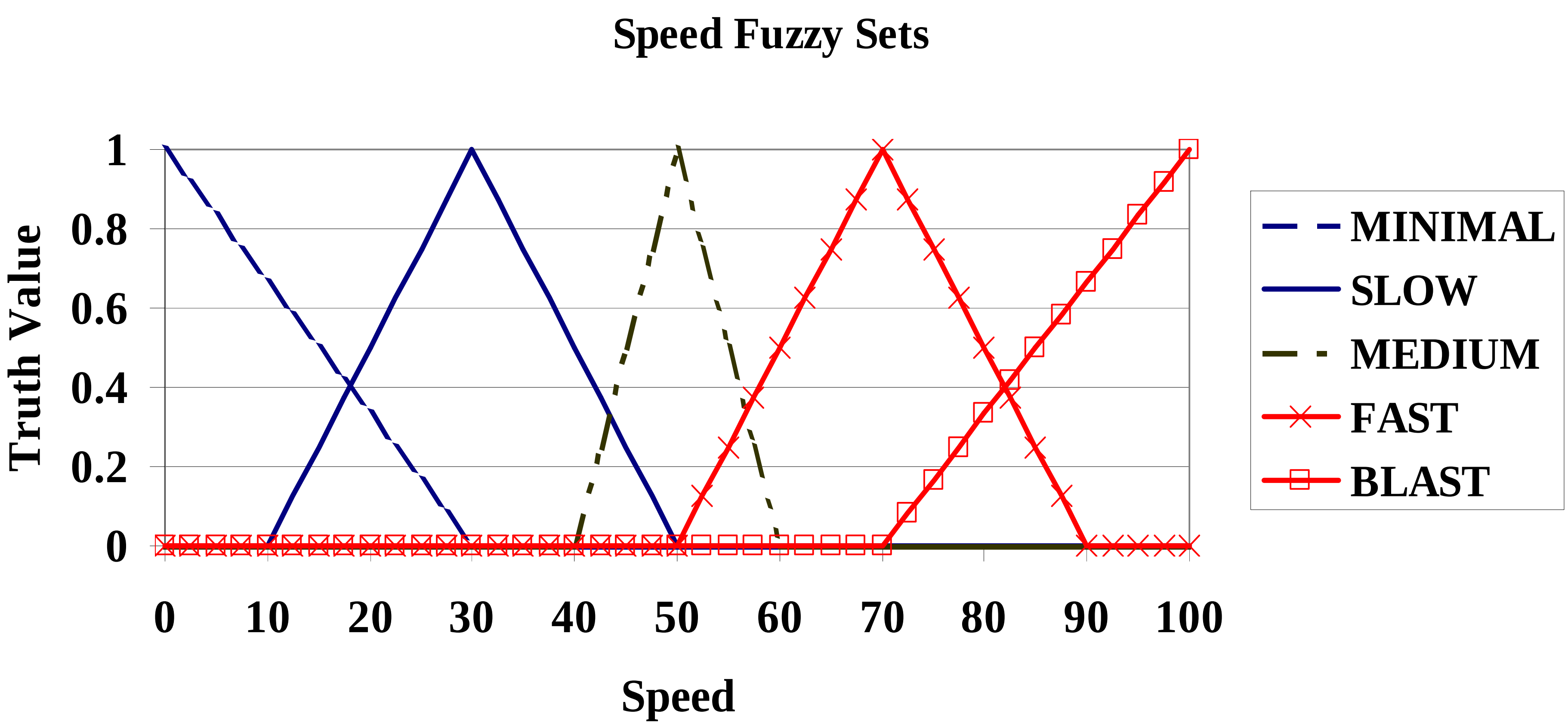
  for $10 \le v \le 30$ $\mu_{SLOW}(t) = v / 20 - 0.5$

  for $30 \le v \le 50$ $\mu_{SLOW}(t) = -v / 20 + 2.5$

- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



Exercises

For

$A = \{0.2/a, 0.4/b, 1/c, 0.8/d, 0/e\}$

$B = \{0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e\}$

Draw the Fuzzy Graph of *A* and *B*

Then, calculate the following:

- Support, Core, Cardinality, and Complement for *A* and *B* independently
- Union and Intersection of A and B
- the new set C, if $C = A^2$
- the new set D, if $D = 0.5 \times B$
- the new set E, for an alpha cut at $A_{0.5}$

Solutions

A = {0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}
B = {0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}

Support

Supp(A) = {a, b, c, d}
Supp(B) = {b, c, d, e}

Core

Core(A) = {c}
Core(B) = {}

Cardinality

Card(A) = 0.2 + 0.4 + 1 + 0.8 + 0 = 2.4
Card(B) = 0 + 0.9 + 0.3 + 0.2 + 0.1 = 1.5

Complement

Comp(A) = {0.8/a, 0.6/b, 0/c, 0.2/d, 1/e}
Comp(B) = {1/a, 0.1/b, 0.7/c, 0.8/d, 0.9/e}

Solutions

$A = \{0.2/a, 0.4/b, 1/c, 0.8/d, 0/e\}$
$B = \{0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e\}$

Union

$A \cup B = \{0.2/a, 0.9/b, 1/c, 0.8/d, 0.1/e\}$

Intersection

$A \cap B = \{0/a, 0.4/b, 0.3/c, 0.2/d, 0/e\}$

$C = A^2$

$C = \{0.04/a, 0.16/b, 1/c, 0.64/d, 0/e\}$

$D = 0.5 \times B$

$D = \{0/a, 0.45/b, 0.15/c, 0.1/d, 0.05/e\}$

$E = A_{0.5}$

$E = \{c, d\}$